

Simplified results on electric resistance on a distance-regular graph.

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Abstract

Simplifications of a result from [MK10] concerning the electric resistance between points in a distance-regular graph are given. In particular, we prove that the maximal resistance between points is bounded by twice the resistance between neighbors. We also show that if the constant is weakened, then a very simple proof can be given.

1 Introduction

In this note, we will consider the electric resistance between points in a graph; that is, we imagine that a graph G is a circuit with each edge representing a wire with unit resistance, and the effective resistance measures the ease with which current moves between points (details will be presented in Section 2). In particular, we are interested in the following result, which was originally conjectured by Biggs in [Big93]:

Theorem 1 *There is a universal constant K such that if G is a distance-regular graph with degree at least 3 and diameter D then*

$$(1) \quad \max_j r_j = r_D \leq K r_1,$$

where r_j is the electric resistance between any two vertices of distance j .

This theorem shows that the class of distance-regular graphs have strong regularity properties with respect to the electric resistance metric. Biggs conjectured further as to the optimal value for K .

Proposition 1 *We may take $K = 1 + \frac{94}{101} \approx 1.931$ in Theorem 1, and equality holds only for the Biggs-Smith graph.*

The result was proved earlier in [MK10], but the proof is rather long and technical, and relies heavily on a library of classification theorems for small distance-regular graphs. One of the purposes of this note is to give a much shorter and simpler proof of this result, using new techniques which were developed in [KMP13] in order to prove the more difficult assertion that $K \searrow 1$ as the degree of the graph goes to ∞ . The new proof is an improvement over the old, but still requires several classification results and a detailed analysis of several different cases. However, if we allow ourselves to accept the worse constant $K = 3$, we will see that there is a very short and simple proof requiring no classification results whatsoever. As this proof is likely to be of more interest to non-specialists than the more difficult ones, and should even be accessible to those with no prior knowledge of distance-regular graphs, we will take the time to present it as well.

Proposition 2 *We may take $K = 3$ in Theorem 1.*

In the next section we introduce the framework which was developed by Biggs in [Big93] which will allow us to prove these results. The ensuing section gives the proofs of Propositions 1 and 2, and we close with a few remarks in the final section.

2 Preliminaries

Let G be a connected graph. To define the effective resistance $r_{u,v}$ between points u, v , we attach a battery of unit voltage between u and v and take the reciprocal of the current which flows through the graph when each edge is taken to have unit resistance. $r_{u,v}$ is then a metric on the graph, and this metric has a large number of important connections to random walks; [DS84] is a highly elegant introduction to this concept. The distance $d(x, y)$ between any two vertices x, y of G is the length of a shortest path between x and y in G . The diameter of G is the maximal distance occurring in G and we will denote this by $D = D(G)$. For a vertex $x \in G$, define $K_i(x)$ to be the set of vertices which are at distance i from x ($0 \leq i \leq D$) where $D := \max\{d(x, y) \mid x, y \in V(G)\}$ is the diameter of G . In addition, define $K_{-1}(x) := \emptyset$ and $K_{D+1}(x) := \emptyset$. We write $x \sim_G y$ or simply $x \sim y$ if two vertices x and y are adjacent in G . A connected graph G with diameter D is called *distance-regular* if there are integers b_i, c_i ($0 \leq i \leq D$) such that for any two vertices $x, y \in V(G)$ with $d(x, y) = i$, there are

precisely c_i neighbors of y in $K_{i-1}(x)$ and b_i neighbors of y in $K_{i+1}(x)$ (cf. [BCN89, p.126]). In particular, any distance-regular graph G is regular with valency $k := b_0$ and we define $a_i := k - b_i - c_i$ for notational convenience. Note that the definition implies that $c_{i+1}|K_{i+1}(x)| = b_i|K_i(x)|$, so that in fact $|K_i(x)| = \frac{b_0 \dots b_{i-1}}{c_1 \dots c_i}$. A straightforward consequence of the definition is that (cf. [BCN89, Proposition 4.1.6])

- (i) $k = b_0 > b_1 \geq \dots \geq b_{D-1}$;
- (ii) $1 = c_1 \leq c_2 \leq \dots \leq c_D$;
- (iii) $b_i \geq c_j$ if $i + j \leq D$.

Henceforth we work entirely with a distance-regular graph G on n vertices with associated intersection array $(b_0, b_1, \dots, b_{D-1}; c_1, \dots, c_D)$, and we assume further that $k = b_0 \geq 3$. The *Biggs potentials* are defined recursively for $0 \leq i \leq D-1$ by

$$(2) \quad \begin{aligned} \phi_0 &= n - 1 \\ \phi_i &= \frac{c_i \phi_{i-1} - k}{b_i} \end{aligned}$$

This recursive definition leads to the explicit value:

$$(3) \quad \phi_i = k \left(\frac{1}{c_{i+1}} + \frac{b_{i+1}}{c_{i+1}c_{i+2}} + \dots + \frac{b_{i+1} \dots b_{D-1}}{c_{i+1} \dots c_D} \right).$$

Note that

$$(4) \quad \begin{aligned} \phi_{i-1} - \phi_i &= k \left(\left(\frac{1}{c_i} - \frac{1}{c_{i+1}} \right) + \left(\frac{b_i}{c_i c_{i+1}} - \frac{b_{i+1}}{c_{i+1} c_{i+2}} \right) + \dots \right. \\ &\quad \left. + \left(\frac{b_i \dots b_{D-2}}{c_i \dots c_{D-1}} - \frac{b_{i+1} \dots b_{D-1}}{c_{i+1} \dots c_D} \right) + \frac{b_i \dots b_{D-1}}{c_i \dots c_D} \right). \end{aligned}$$

Conditions (i) and (ii) above show that this quantity is positive, so ϕ_i is a strictly decreasing sequence. In [Big93](or see [MK10]), the following was shown.

Proposition 3 *The resistance between two vertices of distance j in G is given by*

$$(5) \quad \frac{2 \sum_{0 \leq i < j} \phi_i}{nk}$$

Note that this proposition, together with Proposition 1 and the fact that $\phi_0 = n - 1$, show that

Proposition 4 *The resistance $r_{u,v}$ between any vertices $u, v \in G$ satisfies*

$$(6) \quad r_{u,v} < \frac{4}{k}.$$

In light of Proposition 3, it is clear that Propositions 1 and 2 can be verified by proving that

$$(7) \quad \phi_1 + \dots + \phi_{D-1} < K\phi_0.$$

This is what we will show.

3 Proof of Propositions

We begin by describing the general technique which will be used in both proofs. It is well-known that the case $b_1 = 1$ occurs only for cocktail party graphs, and the results are simple to verify in that case, so we will assume always that $b_1 \geq 2$. It is clear that (2) implies

$$(8) \quad \phi_i < \frac{c_i}{b_i} \phi_{i-1}.$$

This will be very useful to us so long as $b_i > c_i$. At such point as $c_i \geq b_i$, however, it will be more profitable to bound the expression in (3), as (iii) above implies that this occurs when i is relatively close to D , and ϕ_i will therefore be the sum of a small number of small terms. In light of this, we set $j = \inf\{i : c_i \geq b_i\}$. We consider $\phi_0, \phi_1, \dots, \phi_{j-1}$ to be the *head* of the sequence, and ϕ_j, \dots, ϕ_D to be the *tail*. There is another interesting consequences of these definitions. Recall that $|K_i(x)| = \frac{b_0 \dots b_{i-1}}{c_1 \dots c_i}$, where $K_i(x)$ The following lemma will be key for bounding the tail.

Lemma 1 $\phi_j + \dots + \phi_{D-1} \leq (j - 1/2)\phi_{j-1}$.

This lemma is not particularly difficult, and a proof can be found in [KMP13]. For bounding the head, we will simply observe that when $b_i > c_i$ then since $b_i \leq b_1$ and b_i, c_i are integers we have $\frac{c_i}{b_i} < \frac{b_1-1}{b_1}$. In conjunction with (8), we then have

$$(9) \quad \phi_i < \left(\frac{b_1 - 1}{b_1} \right) \phi_{i-1}.$$

Furthermore, since $c_1 = 1$, (8) implies

$$(10) \quad \phi_1 < \frac{1}{b_1} \phi_0.$$

We begin with the easier Proposition 2.

Proof of Proposition 2: To simplify the notation, we set $\alpha = \frac{b_1-1}{b_1}$. We bound the head and tail in (7) as described above, using (9), (10), and Lemma 1, to obtain

$$(11) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{1}{b_1} + \frac{\alpha}{b_1} + \dots + \frac{\alpha^{j-2}}{b_1} + (j-1/2) \frac{\alpha^{j-2}}{b_1}.$$

Note that the term $\frac{\alpha^{j-2}}{b_1}$ corresponds to $\frac{\phi_{j-1}}{\phi_0}$, so that the final term is the bound on $\frac{\phi_j + \dots + \phi_{D-1}}{\phi_0}$ given by Lemma 1. We now replace the head bound by the geometric series

$$(12) \quad \frac{1}{b_1} + \frac{\alpha}{b_1} + \frac{\alpha^2}{b_1} + \dots = \frac{1}{b_1(1-\alpha)} = \frac{1}{b_1(1-\frac{b_1-1}{b_1})} = 1.$$

In order to control the tail term, we set $f(i) = \frac{(i-1/2)\alpha^i}{b_1}$. Note that if $i \geq b_1$ then

$$(13) \quad \frac{f(i+1)}{f(i)} = \frac{b_1-1}{b_1} \times \frac{i+1/2}{i-1/2} \leq \frac{i-1}{i} \times \frac{i+1/2}{i-1/2} < 1,$$

whereas if $i \leq b_1 - 1$ we have

$$(14) \quad \frac{f(i+1)}{f(i)} = \frac{b_1-1}{b_1} \times \frac{i+1/2}{i-1/2} \geq \frac{i}{i+1} \times \frac{i+1/2}{i-1/2} > 1.$$

Thus, $f(i)$ attains its maximum at $i = b_1$. We therefore have

$$(15) \quad (j-1/2) \frac{\alpha^{j-2}}{b_1} \leq (b_1-1/2) \frac{\alpha^{b_1-2}}{b_1} \leq \frac{b_1-1/2}{b_1} < 1.$$

Combining the estimates (11), (12), and (15) gives the proposition. □

Proof of Proposition 1: The proof proceeds by considering a number of separate cases. We will show

$$(16) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} < .93$$

for all graphs other than the Biggs-Smith graph. If $b_1 \leq 2$, then it is known that either $D \leq 2$ or $k \leq 4$. We may therefore reduce our problem to the case $b_1 \geq 3$ by disposing of the following two cases.

Case 1 : $D \leq 2$.

There is nothing to show for $D = 1$, and for $D = 2$ we need only show $\phi_1 < .93\phi_0$. This is clear from (10) and the assumption $b_1 \geq 2$. \square

Case 2 : $k = 3$ or 4 .

The distance-regular graphs of valency 3 and 4 have been classified (see [BCN89, Thm 7.5.1] and [BK99]). The corresponding values of $\frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0}$ are given in the following table, which contains only the graphs with $D \geq 3$. Note that all are less than .93 except for that of the Biggs-Smith graph.

Name	Vertices	Intersection array	$\frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0}$
Cube	8	(3,2,1;1,2,3)	0.428571
Heawood graph	14	(3,2,2;1,1,3)	0.461538
Pappus graph	18	(3,2,2,1;1,1,2,3)	0.588235
Coxeter graph	28	(3,2,2,1;1,1,1,2)	0.666667
Tutte's 8-cage	30	(3,2,2,2;1,1,1,3)	0.655172
Dodecahedron	20	(3,2,1,1,1;1,1,1,2,3)	0.842105
Desargues graph	20	(3,2,2,1,1;1,1,2,2,3)	0.710526
Tutte's 12-cage	126	(3,2,2,2,2,2;1,1,1,1,1,3)	0.872
Biggs-Smith graph	102	(3,2,2,2,1,1,1;1,1,1,1,1,3)	0.930693
Foster graph	90	(3,2,2,2,2,1,1,1,1,1,2,2,2,3)	0.896067
$K_{5,5}$ minus a matching	10	(4,3,1;1,3,4)	0.296296
Nonincidence graph of $PG(2, 2)$	14	(4,3,2;1,2,4)	0.307692
Line graph of Petersen graph	15	(4,2,1;1,1,4)	0.428571
4-cube	16	(4,3,2,1;1,2,3,4)	0.422222
Flag graph of $PG(2, 2)$	21	(4,2,2;1,1,2)	0.5
Incidence graph of $PG(2, 3)$	26	(4,3,3;1,1,4)	0.32
Incidence graph of $AG(2, 4)$ -p.c.	32	(4,3,3,1;1,1,3,4)	0.376344
Odd graph O_4	35	(4,3,3;1,1,2)	0.352941
Flag graph of $GQ(2, 2)$	45	(4,2,2,2;1,1,1,2)	0.681818
Doubled odd graph	70	(4,3,3,2,2,1,1,1,1,2,2,3,3,4)	0.521739
Incidence graph of $GQ(3, 3)$	80	(4,3,3,3;1,1,1,4)	0.417722
Flag graph of $GH(2, 2)$	189	(4,2,2,2,2,2;1,1,1,1,1,2)	0.882979
Incidence graph of $GH(3, 3)$	728	(4,3,3,3,3,3;1,1,1,1,1,4)	0.485557

□

□

Case 3 : $b_1 \geq 3, c_2 = 1$.

If $j = 2$, then applying Lemma 1 gives $\phi_1 + \dots + \phi_{D-1} \leq \frac{5}{2}\phi_1$. As $\phi_1 \leq \frac{\phi_0}{b_1} \leq \frac{\phi_0}{3}$, we get

$$(17) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{5}{6} < .93$$

If $j = 3$, then $b_2 \geq 2$, hence $\phi_2 \leq \frac{\phi_1}{2} \leq \frac{\phi_0}{6}$. By Lemma 1 we have

$$(18) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{\phi_0/3 + (7/2)\phi_0/6}{\phi_0} = \frac{11}{12} < .93$$

If $j \geq 4$ and $b_2 = 2$ then we must have $b_3 = 2, c_3 = 1$, so that $\frac{b_2 b_3}{c_2 c_3} = 4$. On the other hand, if this does not occur than $\frac{b_2}{c_2} \geq 3$. We will consider these cases separately.

Subcase 1: $\frac{b_2}{c_2} \geq 3$.

For $i < j$ we have $b_1 \geq b_i > c_i$, and for any i with $c_i > 1$ we must have $b_i < b_1$, by Proposition 5.4.4 in [BCN89]. Thus, $\frac{c_i}{b_i} \leq \frac{b_1-2}{b_1-1}$. Define $\alpha = \frac{b_1-2}{b_1-1}$. Applying Lemma 1 we have

$$(19) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{1}{b_1} + \frac{1}{3b_1} + \frac{\alpha}{3b_1} + \dots + \frac{\alpha^{j-3}}{3b_1} + \frac{(j-1/2)\alpha^{j-3}}{3b_1}$$

Replace the second through $(j-1)$ th term by a geometric series to obtain

$$(20) \quad \begin{aligned} \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} &< \frac{1}{b_1} + \frac{1}{3b_1} \left(\frac{1}{1 - \frac{b_1-2}{b_1-1}} \right) + \frac{(j-1/2)\alpha^{j-3}}{3b_1} \\ &= \frac{1}{b_1} + \frac{b_1-1}{3b_1} + \frac{(j-1/2)\alpha^{j-3}}{3b_1}. \end{aligned}$$

Using the same technique as in the proof of Proposition 2, set $f(i) = \frac{(i-1/2)\alpha^{i-3}}{3b_1}$. Note that if $i \geq b_1 - 1$ then

$$(21) \quad \frac{f(i+1)}{f(i)} = \frac{b_1-2}{b_1-1} \times \frac{i+1/2}{i-1/2} \leq \frac{i-1}{i} \times \frac{i+1/2}{i-1/2} < 1,$$

whereas if $i \leq b_1 - 2$ we have

$$(22) \quad \frac{f(i+1)}{f(i)} = \frac{b_1-2}{b_1-1} \times \frac{i+1/2}{i-1/2} \geq \frac{i}{i+1} \times \frac{i+1/2}{i-1/2} > 1$$

Thus, $f(i)$ attains its maximum at $i = b_1 - 1$. Using this to bound the final term in (20) for $b_1 \geq 4$ gives

$$(23) \quad \begin{aligned} \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} &< \frac{1}{b_1} + \frac{b_1-1}{3b_1} + \frac{(b_1-3/2)\alpha^{b_1-4}}{3b_1} \\ &\leq \frac{1}{b_1} + \frac{b_1-1}{3b_1} + \frac{1}{3} = \frac{2b_1+2}{3b_1} < .93 \end{aligned}$$

If $b_1 = 3$, then since $j \geq 4$ we can simply plug in $j = 4$ to get

$$(24) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} < \frac{1}{b_1} + \frac{b_1 - 1}{3b_1} + \frac{(7/2)\alpha}{3b_1} \\ = \frac{3}{4} < .93$$

Subcase 2: $\frac{b_2 b_3}{c_2 c_3} \geq 4$.

This follows much as in the previous case. Let $\alpha = \frac{b_1 - 2}{b_1 - 1}$. Since $b_2 \geq b_3$ and $c_2 \leq c_3$ we must have $\frac{b_2}{c_2} \geq 2$. We then have

$$(25) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{1}{b_1} + \frac{1}{2b_1} + \frac{1}{4b_1} + \frac{\alpha}{4b_1} + \dots + \frac{\alpha^{j-4}}{4b_1} + \frac{(j - 1/2)\alpha^{j-4}}{4b_1}.$$

If $j = 4$, then in fact the terms containing α 's are not present, and we get

$$(26) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{1}{b_1} + \frac{1}{2b_1} + \frac{1}{4b_1} + \frac{(4 - 1/2)}{4b_1} = \frac{10\frac{1}{2}}{4b_1} \leq \frac{10\frac{1}{2}}{12} < .93.$$

If $j > 4$, then we again set $f(i) = \frac{(i-1/2)\alpha^{i-4}}{4b_1}$, and use the argument from the previous subcase to conclude that $f(i)$ is decreasing for $i \geq b_1 - 1$ but increasing for $i < b_1 - 1$. If $b_1 < 6$, we may therefore replace j by 5 in (25) and sum the geometric series in α to get a bound of

$$(27) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{3}{2b_1} + \frac{1}{4} \left(\frac{b_1 - 1}{b_1} \right) + \frac{(4\frac{1}{2})\alpha}{4b_1}.$$

If $b_1 = 4, 5$, then this expression is seen to be less than .93 upon replacing α by 1, while for $b_1 = 3$ a sufficient bound is obtained by using $\alpha = \frac{1}{2}$. If $b_1 \geq 6$, we can replace j by $b_1 - 1$ in (25) and again sum the geometric series to obtain

$$(28) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{3}{2b_1} + \frac{1}{4} \left(\frac{b_1 - 1}{b_1} \right) + \frac{(b_1 - 3/2)\alpha^{b_1-5}}{4b_1} < \frac{3}{2b_1} + \frac{1}{2} < .93.$$

□

Case 4 : $b_1 \geq 3, j = 3, c_2 > 1$.

By Theorem 5.4.1 in [BCN89], $c_2 \leq \frac{2}{3}c_3$. Suppose first that $c_3 > b_3$; then $D \leq 2j - 1 = 5$ by Property (iii) in Section 2, and $\phi_2 < \frac{b_1-1}{b_1}\phi_1 < \frac{b_1-1}{b_1^2}\phi_0$. We therefore have

$$(29) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} < \frac{1}{b_1} + 3\left(\frac{b_1-1}{b_1^2}\right) = \frac{4b_1-3}{b_1^2}.$$

For $b_1 \geq 4$, this is less than .93. If $b_1 = 3$, then we must have $c_2 = 1$, since if $c_2 > 1$ then $b_2 < 3$ by Proposition 5.4.4 in [BCN89], contradicting $j = 3$. Thus, $\phi_2 < \frac{1}{2}\phi_1 < \frac{1}{6}\phi_0$, and we obtain

$$(30) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} < \frac{1}{3} + 3 \times \frac{1}{6} < .93.$$

If $c_3 = b_3 \leq b_2$, then if we assume $\frac{c_2}{b_2} \leq \frac{1}{2}$ by Lemma 1 we have

$$(31) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{\phi_1 + (7/2)\phi_2}{\phi_0} \leq \frac{1}{b_1} + \frac{7}{4b_1} = \frac{11}{4b_1} < .93$$

On the other hand, if it is not the case that $\frac{c_2}{b_2} \leq \frac{1}{2}$, then the proof of Theorem 5.4.1 of [BCN89] implies that G contains a quadrangle. By Corollary 5.2.2 in [BCN89], $D \leq \frac{2k}{k+1-b_1}$. It is straightforward to verify that the fact that $k \geq b_1 + 1$ implies that

$$(32) \quad \frac{2k}{k+1-b_1} \leq b_1 + 1$$

We therefore see that the fact that G contains a quadrangle implies $D \leq b_1 + 1$. Furthermore, we still have $\frac{c_2}{b_2} \leq \frac{2}{3}$ by Theorem 5.4.1 of [BCN89]. We therefore have

$$(33) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{\phi_1 + (b_1-1)\phi_2}{\phi_0} \leq \frac{1}{b_1} + \frac{2(b_1-1)}{3b_1} = \frac{2b_1+1}{3b_1} < .93$$

□

Case 5 : $b_1 \geq 3, j \geq 4, c_2 > 1, G$ contains a quadrangle.

As in the argument given in Case 4, we see that G containing a quadrangle implies $D \leq b_1 + 1$. Furthermore, Theorem 5.4.1 of [BCN89] implies that $c_3 \geq (3/2)c_2$. Since $j \geq 4$ and thus $b_2 \geq b_3 > c_3$ we must have $\frac{c_2}{b_2} \leq \frac{2}{3}$. This gives

$$(34) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} \leq \frac{1}{b_1} + (b_1 - 1) \frac{2}{3b_1} = \frac{2b_1 + 1}{3b_1} < .93$$

□

Case 6 : $b_1 \geq 3, j \geq 4, c_2 > 1$, G does not contain a quadrangle.

In this case G is a Terwilliger graph. By Corollary 1.16.6 of [BCN89], if $k < 50(c_2 - 1)$ then $D \leq 4$, which was covered in [Big93]. Thus, we can assume $k \geq 50(c_2 - 1) \geq 50$, which implies $b_1 \geq \frac{k}{2} > 20$. By Theorem 3 of [KMP13],

$$(35) \quad \phi_2 + \dots + \phi_{D-1} < 9\phi_1$$

We then have

$$(36) \quad \frac{\phi_1 + \dots + \phi_{D-1}}{\phi_0} < \frac{10\phi_1}{\phi_0} < \frac{10(\frac{\phi_0}{20})}{\phi_0} = \frac{1}{2}$$

□

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